Extended self-similarity and hierarchical structure in turbulence

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We show that a generalization of the She-Leveque hierarchical structure [Z.S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994)] together with a constant maximum magnitude of the velocity difference give rise to the extended self-similarity (ESS) [R. Benzi *et al.*, Phy. Rev. E **48**, R29 (1993)]. Our analysis thus suggests that the ESS measured in turbulent flows is an indication of the most intense structures being shocklike. Analyses of velocity measurements in a turbulent pipe flow support our conjecture.

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Fully developed turbulence is characterized by power-law dependence of the moments of velocity fluctuations. Kolmogorov in 1941 ($K41$) [1] predicted that the velocity structure functions $S_p(r)$ have simple power-law dependence on r within the inertial range

$$S_p(r) \equiv \langle |\delta v_r|^p \rangle \sim \epsilon^{p/3} r^{p/3}. \tag{1}$$

Here $\delta v_r = v(x+r) - v(x)$ is the velocity difference across a separation r. Experiments [2] have indicated that there is indeed power-law scaling in the inertial range but the scaling exponents are different from $p/3$. That is,

$$S_p(r) \sim r^{\zeta_p} \tag{2}$$

and ζ_p has a nonlinear dependence on p. This implies that the functional form of the probability density function of δv_r depends on r, that is, the velocity fluctuations have scale-dependent statistics. Understanding this deviation from $K41$, the anomalous scaling, is essential to our fundamental understanding of the small scale statistical properties of turbulence.

Recently, Benzi *et al.* [3] have discovered a remarkable new scaling property: $S_p(r)$ has a power-law dependence on $S_3(r)$ over a range substantially longer than the scaling range obtained by plotting $S_p(r)$ as a function of r. This behavior was named *extended self-similarity* (ESS). Its discovery has enabled more accurate determination of the relative scaling exponents, particularly at moderately high Reynolds numbers accessible experimentally and numerically. It was later reported that such an extension is limited for anisotropic turbulent flows such as atmospheric boundary layer and channel flow [4–6]. This inspires the study of a generalized ESS (GESS), a scaling behavior of the normalized structure functions when plotted against each other [7,8], which is still valid in these anisotropic flows. The validity of ESS suggests that the different order structure functions have the same dependence on r, when r is near the dissipative range [9–11]. Very recently, Yakhot argued that some mean-field approximation of the pressure contributions in the Navier-Stokes equation would lead to ESS [12].

To account for the anomalous scaling exponents ζ_p, She and Leveque [13] have proposed a model of hierarchical structure (HS). When stated for the velocity structure functions, the HS hypothesis reads

$$\frac{S_{p+2}(r)}{S_{p+1}(r)} = A_{p+1} \left[\frac{S_{p+1}(r)}{S_p(r)} \right]^{\beta} [S^{(\infty)}(r)]^{1-\beta}. \tag{3}$$

Here $S^{(\infty)}(r) \equiv \lim_{p\to\infty} S_{p+1}(r)/S_p(r)$ and $0 < \beta < 1$ is a constant. This hypothesis (and the similar version for the local energy dissipation) was supported by experimental velocity measurements taken in turbulent jets and wakes [14–16]. Moreover, the passive scalar structure functions [17] and the local passive scalar dissipation [18], the temperature structure functions [19] and the local temperature dissipation [20] in turbulent convection, and the velocity structure functions in a class of shell models [21–23] were all found to possess similar hierarchical structures.

The HS description advocates the importance of the quantity $S^{(\infty)}(r)$. If the probability density function of δv_r is finite and is zero beyond a certain value, one can show that $S^{(\infty)}(r)$ is equivalent to $\lim_{p\to\infty} S_p^{1/p}$, and is equal to $|\delta v_r|^{\max}$, the maximum magnitude of the velocity difference. Thus, in any flows in which the velocity is bounded, $S^{(\infty)}(r)$ would exist, and is associated with the most intense structures of the flow.

In this paper, we show that the She-Leveque hierarchical structure (SLHS) leads to GESS. In other words, SLHS is a special form of GESS. We then give a generalization of SLHS, which is equivalent to GESS, with the constant β replaced by a function of p. Furthermore, we show that if $|\delta v_r|^{\max}$ is independent of r for r within the GESS range, the generalized SLHS (and hence GESS) will give rise to ESS. This leads to our conjecture that the observed ESS in turbulent flows is an indication of the most intense structures being shocklike with a constant $|\delta v_r|^{\max}$. We note that in the Burgers equation, the presence of shocks will give a $|\delta v_r|^{\max}$ that is independent of r. Thus, we use the term shocklike, which is meant to be indicative only. Consequently, we predict that in anisotropic flows where GESS but not ESS holds, $|\delta v_r|^{\max}$ has a r dependence. Finally, we note that in any finite number of measurements, the detectable $|\delta v_r|^{\max}$ would depend on the sample size. It is thus important to have a procedure that allows one to study the true $|\delta v_r|^{\max}$ or $S^{(\infty)}(r)$. We develop such a procedure, a systematic HS analysis, for analyzing experimental turbulent signals. Application of the HS analysis to turbulent velocity fluctuations in a pipe flow demonstrates that $|\delta v_r|^{\max}$ indeed depends on r in the near-wall strong-shear regions where only GESS but not ESS is valid, and is consistent with being r independent in the centerline of the pipe where ESS is valid.

Rewrite Eq. (3) as

$$\frac{S_{k+1}(r)}{S_k(r)S^{(\infty)}(r)}=A_k\left[\frac{S_k(r)}{S_{k-1}(r)S^{(\infty)}(r)}\right]^{\beta}, \tag{4}$$

for integer k, which implies

$$S_n(r)=B_n\left[\frac{S_1(r)}{S^{(\infty)}(r)}\right]^{(1-\beta^n)/(1-\beta)}[S^{(\infty)}(r)]^n, \tag{5}$$

where $B_n\equiv\Pi_{k=1}^{n}\Pi_{j=0}^{k-1}A_j^{\beta^{k-1-j}}$. Equation (5) then gives rise to

$$T_n(r)\sim T_m(r)^{\rho(n,m)}, \tag{6}$$

which is a scaling behavior for the normalized structure functions, $T_n(r)\equiv S_n(r)/S_3(r)^{n/3}$, with the normalized exponents $\rho(n,m)$, even when $S_n(r)$ does not have a scaling behavior in r. For SLHS,

$$\rho(n,m)=\frac{3(1-\beta^n)-n(1-\beta^3)}{3(1-\beta^m)-m(1-\beta^3)}. \tag{7}$$

We have thus proved that SLHS implies GESS.

In Ref. [8], Benzi *et al.* interpreted GESS in terms of a random multiplicative process, and, thus, also touched upon the possible relation between GESS and SLHS. They showed that GESS holds if

$$S_p(r)\sim g_1(r)^p g_2(r)^{H(p)} \tag{8}$$

for any functions $g_1(r)$, $g_2(r)$, and $H(p)$. If Eq. (6) is valid, we can always write the structure functions in the form of Eq. (8), say, with $g_1(r)=S_3(r)^{1/3}$, $g_2(r)=S_{q*}(r)/S_3(r)^{q*/3}$, and $H(p)=\rho(p,q*)$ for any chosen value of $q*$. Thus, Eq. (8) is actually equivalent to GESS.

Our demonstration here gives the functions $g_1(r)$ and $g_2(r)$ a meaning. Although the choice of $g_1(r)$ and $g_2(r)$ is not unique, the following forms can be obtained from Eq. (5): $g_1(r)=S^{(\infty)}(r)$ and $g_2(r)=S_3(r)/[S^{(\infty)}(r)]^3$. Here, $g_1(r)$ describes the r dependence of the strongest fluctuations $S^{(\infty)}(r)$ and $g_2(r)$ describes the normalized r dependence of (typical) weak fluctuations [e.g., $S_3(r)$] by $[S^{(\infty)}(r)]^3$. Expressing g_1 and g_2 this way, we have

$$S_p(r)\sim[S^{(\infty)}(r)]^p\left\{\frac{S_3(r)}{[S^{(\infty)}(r)]^3}\right\}^{f(p)}, \tag{9}$$

where $f(p)$ is a function to be discussed below. Note that when $g_2(r)$ is constant, or the weak fluctuations have the same r dependence as the strongest fluctuations, we have the $K41$ simple scaling. The function $f(p)$ has to satisfy various conditions. By definition, $f(0)=0$, $f(3)=1$, and $\lim_{p\to\infty}f(p+1)-f(p)=\lim_{p\to\infty}f(p)/p=0$. Furthermore, the boundedness of the velocity restricts ζ_p to be a nondecreasing function of p [2]. This is guaranteed if $df(p)/dp\geqslant 0$.

Note that Eq. (9) is a general expression of GESS. Rewriting Eq. (9) in the form of Eq. (4), we have

$$\frac{S_{p+1}(r)}{S_p(r)S^{(\infty)}(r)}=C_p\left[\frac{S_p(r)}{S_{p-1}(r)S^{(\infty)}(r)}\right]^{g(p)}, \tag{10}$$

where $g(p)\equiv[f(p+1)-f(p)]/[f(p)-f(p-1)]$. It is clear that SLHS corresponds to the particular case of $g(p)=\beta$ or equivalently

$$f(p)=\frac{1-\beta^p}{1-\beta^3}. \tag{11}$$

We list two additional possible classes of $f(p)$: $f(p)=[(p\sigma+1)^{\alpha}-1]/[(3\sigma+1)^{\alpha}-1]$ and $f(p)=\ln(p\sigma+1)/\ln(3\sigma+1)$ with $\sigma>0$ and $0<\alpha<1$. The latter is the limit of the former when $\alpha\to 0$. These two cases correspond to the two results derived by Novikov, Eqs. (18) and (19) in Ref. [24], using the theory of infinitely divisible distributions.

To study when ESS would also be valid, we rewrite Eq. (6) in the following form:

$$\frac{S_p(r)^{1/p}}{S_3(r)^{1/3}}\sim\left[\frac{S_q(r)^{1/q}}{S_3(r)^{1/3}}\right]^{(q/p)\rho(p,q)} \tag{12}$$

with

$$\rho(p,q)=\frac{f(p)-p/3}{f(q)-q/3}. \tag{13}$$

Thus, we see that if there exists a $p*\neq 0$ such that $S_{p*}(r)^{1/p*}$ is independent of r, then we would have a scaling behavior of S_p vs S_3 (and thus S_q for $q\neq p$). If $p*$ is finite, $\zeta_{p*}=0$, which implies $\zeta_m=0$ for $0\leqslant m\leqslant p*$ because ζ_n has to be a nondecreasing function of n. It further gives $\zeta_p=0$ for all values of p if ζ_p is an analytic function of p. This could be avoided only if $p*\to\infty$. Hence, GESS together with the condition that $S^{(\infty)}(r)$ is independent of r would give rise to ESS,

$$S_p(r)\sim S_3(r)^{\eta(p,3)}, \tag{14}$$

with $\eta(p,3)=f(p)$, even when $S_p(r)$ does not have a power-law dependence on r.

Using Eq. (9), an independence of $S^{(\infty)}$ on r implies

$$\zeta_p=\zeta_3 f(p). \tag{15}$$

For SLHS, $\lim_{p\to\infty}f(p)=1/(1-\beta^3)$ hence Eq. (15) further implies a saturation of ζ_p as $p\to\infty$. A connection of the saturation of the exponents with the existence of shocks was suggested earlier by Chen and Cao [25].

Both ESS and SLHS have been observed in a variety of turbulent flow fields. The above analysis points to the interesting conjecture that the two combined is an indication of a property of the flow field that $|\delta v_r|^{\max}$ is independent of r, and the corresponding physical picture is that the most intense structures are shocklike. One way to check the plausibility of these ideas is to examine experimental velocity measurements that demonstrate GESS, and to investigate whether $|\delta v_r|^{\max}$ is indeed independent of r when ESS is valid and otherwise dependent on r when ESS is not valid.

As the detectable $|\delta v_r|^{\max}$ in any finite number of measurements would almost surely underestimate its true value, we have instead developed a systematic procedure to get an indirect estimate of the r dependence of $S^{(\infty)}(r)$. This procedure is derived from the HS model, and we call it a HS analysis. It works when GESS is of the special form of SLHS.

One first verifies if SLHS is valid by performing a β test. It consists of computing $T_p(r)$ and obtaining $\rho(p,q)$ by measuring the slopes of $\ln(T_p)$ vs $\ln(T_q)$. Let $\Delta\rho(p,q) = \rho(p+1,q) - \rho(p,q)$. It is easy to derive the following equation when SLHS is valid:

$$\Delta\rho(p+1,q) = \beta\Delta\rho(p,q) + \frac{(1-\beta)(1-\beta^3)}{q(1-\beta^3) - 3(1-\beta^q)}. \quad (16)$$

If one finds parallel straight lines when plotting $\Delta\rho(p+1,q)$ vs $\Delta\rho(p,q)$ for a set of values of q, we say that the data pass the β test and the turbulent flow field possesses the SLHS property. The slope and intercept provide a double estimate of the constant β. With the estimated β, one can then construct $f(p)$ using Eq. (11).

For an indirect estimate of $S^{(\infty)}(r)$, we compute

$$F_p(r,r_0) \equiv \frac{\ln[S_p(r)/S_p(r_0)] - f(p)\ln[S_3(r)/S_3(r_0)]}{p - 3f(p)} \quad (17)$$

for a fixed value of r_0 within the GESS range, and plot it as a function of r for a set of values of p. From Eq. (9), one sees that $F_p(r,r_0)$ should be independent of p and equal to $\ln[S^{(\infty)}(r)/S^{(\infty)}(r_0)]$. In particular, if $S^{(\infty)}$ is independent of r, then $F_p(r,r_0) = 0$ for r within the GESS range.

We have applied the above procedure to analyze hot-wire measurements of longitudinal velocity fluctuations in a pipe flow of air [26]. The pipe is 22.5 m long with an inner diameter D of 10.5 cm, and the Reynolds number UD/ν is about 1.35×10^5 using the mean velocity along the centerline $U \approx 19.3$ m/s and the kinematic viscosity of air $\nu = 1.5\times10^{-5}$ m^2/s. The velocity measurements were taken as a function of time at 18.2 m away from the entrance both at the centerline of the pipe and at a distance of 0.1 mm from the pipe wall. The estimated wall unit is $y^+ \sim 0.01$ mm, so the near-wall measurements were taken at about ten wall units, where we expect to see a turbulence with strong shear. We have computed the velocity structure functions using the standard Taylor's frozen flow hypothesis. Thus the separation

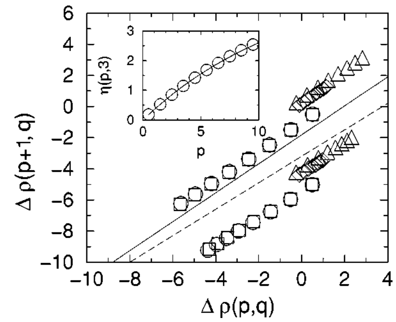

FIG. 1. $\Delta\rho(p+1,q)$ vs $\Delta\rho(p,q)$ for $q=1$ (squares), $q=2$ (circles), $q=4$ (triangles), and $q=5$ (diamonds). Above the solid line (slope$=0.93$) are the centerline data and below the dashed line (slope$=0.85$) are the near-wall data (shifted for clarity). In the inset, we see a good agreement of $\eta(p,3)$ (circles) with $f(p)$ (solid line) for the centerline data.

r is measured in units of the sampling time 1/48 ms using $r = U\tau$, where τ is the time separation. All r's quoted below will be in the same units. The maximum order computed is $p=10$, which is possible because of the large number of measurements taken at each point$=5.76\times10^7$. We have checked that GESS is valid for both sets of measurements but ESS is only valid at the centerline. The range of GESS (and also ESS when valid) is $r \approx 10-500$.

We have obtained $\rho(p,q)$ for the two locations and $\eta(p,3)$ for the centerline location. In Fig. 1, we plot $\Delta\rho(p+1,q)$ vs $\Delta\rho(p,q)$ for some values of q for both locations. The data can be fitted by parallel straight lines showing that the HS is indeed of the She-Leveque form at both locations. We estimate the value of β simultaneously from the slope and the intercept and get 0.93 ± 0.01 and 0.85 ± 0.01, respectively, for the centerline and the near-wall location. In the inset, we see good agreement of $\eta(p,3)$ with $f(p)$ for the centerline measurements. This agreement is an indication of the r independence of $S^{(\infty)}$, which is in accord with the validity of ESS in these measurements.

We have next computed $F_p(r,r_0)$ with $r_0=69$ for both locations and found that the data indeed collapse when p is large. In Fig. 2, we plot $F_p(r,r_0)$ as a function of r. We see that $F_p(r,r_0)$ for the centerline measurements is almost zero for $r \geq 30$ while that for the near-wall measurements shows a

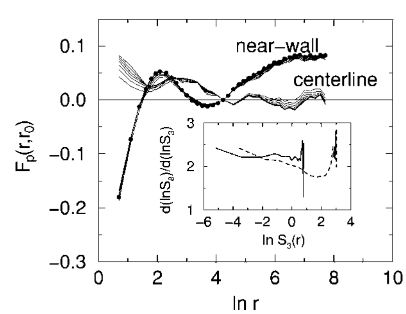

FIG. 2. $F_p(r,r_0)$ for $p \geq 4$ for both the centerline and near-wall measurements (solid lines with filled dots). In the inset, we show the local slope of $\ln S_8(r)$ vs $\ln S_3(r)$ as a function of $\ln S_3(r)$, illustrating that ESS is valid for the centerline (solid line) but not the near-wall (dashed line) measurements.

clear r dependence throughout. Since ESS is valid for the centerline but not for the near-wall measurements (see the inset), these results support our conjecture that the validity of ESS in turbulent flows is an indication of $|\delta v_r|^{\max}$ being r independent.

In summary, we have proposed a link between a measurable statistical property of turbulence, the ESS, and a property of the most intense structures, the r independence of $|\delta v_r|^{\max}$ or $S^{(\infty)}(r)$. We have developed a systematic procedure to test this conjecture when SLHS holds. The analyses of experimental pipe flow data support our conjecture: the near-wall strong-shear turbulence contains more complex most intense structures and that ESS is not valid while the centerline fully developed turbulence has shocklike most intense structures in that $S^{(\infty)}(r)$ is independent of r and ESS is valid. A consequence of the r independence of $S^{(\infty)}(r)$ when SLHS holds is the saturation of the scaling exponents of the velocity structure functions at very high orders. Further experimental and numerical tests are highly desirable.

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